Kuwait University
Math. & Comp.Sci. Dept.

Math. 101
First Exam.

Date: 18/7/1998
Time: 75 min.

Calculators and Mobile Phones are not allowed.

Answer the following questions. Each question counts 5 points

1) Use ϵ and δ definition of the limit to show that

$$\lim_{x\to\frac{1}{2}}(-6x+1)=-2.$$

2) Evaluate the following limit, if it exists

$$\lim_{x\to 0}\frac{4x+2\tan x}{6x-\sin 2x}$$

3 Find the values of x at which the function $f(x)$ is discontinuous, where

$$f(x)=\begin{cases}\dfrac{(x-4)}{\sqrt{x}-2}, & x>0\\[2ex] \dfrac{|x+3|}{x+3}, & x\le 0.\end{cases}$$

Classify the types of discontinuity of f as removable, jump, or infinite.

4) Find the vertical and horizontal asymptotes (if any) of the function

$$f(x)=\frac{x|x-2|}{x^2-5x+6}$$

5) Use the definition of the derivative to find $f'(4)$, where

$$f(x)=\sqrt{x-3}.$$

6) show that the equation $3x^3-5x^2+13x+5=0$ has at least one real root.

7) Find the equation of the tangent line to the graph of

$$f(x)=\frac{\cos^3 x}{\sqrt{x^2+1}}$$

at $x=0$

8) Find the points at which the graph of the function $f(x)=x(x-1)^{\frac{2}{3}}$ has a cusp or a vertical tangent (if any)

GOOD LUCK

Q1.

$\forall \in > 0$, let $|f(x) - L| < \in \longleftrightarrow |(-6x+1)-(-2)| < \in$

$|-6x+1+2| < \in \longleftrightarrow |-6x+3| < \in$

$|-6(x-1/2)| < \in \longleftrightarrow 6|x-1/2| < \in$

$|x-1/2| < \frac{\in}{6} > 0$ put $\delta = \in/6 > 0$

hence $\forall \in > 0 \ \exists \ \delta = \frac{\in}{6} > 0$ s.t $|x-1/2| < \delta$

when ever $|f(x) - (-2)| < \in$ i.e the limit is true.

then $\lim\limits_{x-1/2} (-6x+1) = -2$

Q2.

$$\lim_{x\to 0} \frac{4x + 2\tan x}{6x - \sin 2x} = \lim_{x\to 0} \frac{\frac{4x}{x} + \frac{2\tan x}{x}}{\frac{6x}{x} - \frac{\sin x}{x}} = \frac{\lim\limits_{x\to 0} 4 + 2\lim\limits_{x\to 0} \frac{\tan x}{x}}{\lim\limits_{x\to 0} 6 - 2\lim\limits_{x\to 0} \frac{\sin 2x}{2x}}$$

$$= \frac{4+2}{6-2} = \frac{6}{4} = \frac{3}{2}$$

Q3.

$$\lim_{x\to 4+} f(x) = \lim_{x\to 4+} \frac{(x-4)}{\sqrt{x}-2} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2} = \lim_{x\to 4+} \frac{\cancel{(x-4)}(\sqrt{x}+2)}{\cancel{(x-4)}} = 4$$

$\lim\limits_{x\to 4-} f(x) = 4$, $\lim\limits_{x\to 4} f(x) = 4$

$f(4)$ D.N.E

then f is discont at $x=4$ f has removable discont at $x=4$

• $\lim\limits_{x\to -3^+} f(x) = \lim\limits_{x\to -3^+} \frac{(x+3)}{x+3} = 1$,

• $\lim\limits_{x\to -3^-} f(x) = \lim\limits_{x\to -3^-} \frac{-(x+3)}{x+3} = -1$

$\lim\limits_{x\to -3^+} f(x) \neq \lim\limits_{x\to -3^-} f(x)$ $\therefore \lim\limits_{x\to -3} f(x)$ D.N.E

$f(-3)$ D.N.E

f is discont at $x=-3$, f has a jump discont.

• $\lim\limits_{x\to 0^+} f(x) = 2$, $\lim\limits_{x\to 0^-} f(x) = 1$, $f(0) = 1$

f is discont. at $x=0$ f has a jump discont.

Q4.

$$f(x) = \begin{cases} \dfrac{x\,(x-2)}{(x-2)(x-3)} & x > 2 \\[2ex] \dfrac{-x\,(x-2)}{(x-2)(x-3)} & x < 2 \end{cases}$$

$$= \begin{cases} \dfrac{x}{x-3} & x > 2 \\[2ex] \dfrac{-x}{x-3} & x < 2 \end{cases}$$

- H.A : $y = \lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} \dfrac{x}{x-3} = 1$

$y = \lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} \dfrac{-x}{x-3} = -1$

$y = \pm 1$ are H.A

- V.A : $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} \dfrac{x}{(x-3)} = -2$

$\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} \dfrac{-x}{(x-3)} = 2$

$\lim\limits_{x \to 3} f(x) = \lim\limits_{x \to 3} \dfrac{x|x-2|}{(x-3)(x-2)} = \infty$ ----> $\therefore$ $x = 3$ is V.A

Q5.

$$f'(4) = \lim_{x \to 4} \frac{f(x) - f(4)}{x-4} = \lim_{x \to 4} \frac{\sqrt{x-3} - 1}{x-4}$$

$$= \lim_{x \to 4} \frac{\sqrt{x-3} - 1}{(x-4)} \cdot \frac{\sqrt{x-3} + 1}{\sqrt{x-3} + 1}$$

$$= \lim_{x \to 4} \frac{x-3-1}{(x-4)(\sqrt{x-3} + 1)}$$

$$= 1/2$$

Q8.

Let $f(x) = 3x^3 - 5x^2 + 13x + 5$

$f(0) = 5 > 0$ $\quad f(-1) = -3 - 5 - 13 + 5 = -16 < 0$

$\therefore f(x)$ is cont. on $[-1, 0)$

$f(0) \cdot f(-1) < 0$ By I.V.T $\exists$ at least no $c \in (-1, 0)$

then the equation has at least one real root.

Q7.

$f(0) = \frac{1}{1} = 1$

$$f'(x) = \frac{3\cos^2 x \cdot -\sin x \cdot \sqrt{x^2+1} - \frac{2x}{\sqrt{x^2+1}} \cos^3 x}{(x^2+1)}$$

$f'(0) = 0$ then the T.L is H

The equation of T.L is $y = 1$

Q8.

$D_f = \mathbb{R}$

$$f'(x) = (x-1)^{2/3} + x \cdot 2/3\,(x-1)^{-1/3} = (x-1)^{2/3} + \frac{2x}{3(x-1)^{1/3}}$$

$$= \frac{3(x-1) + 2x}{3(x-1)^{1/3}} = \frac{3x - 3 + 2x}{3(x-1)^{1/3}} = \frac{(5x-3)}{3(x-1)^{1/3}}$$

$$\lim_{x \to 1^+} f'(x) = \lim_{x \to 1^+} \frac{(5x-3)}{3(x-1)^{1/3}} = +\infty$$

$$\lim_{x \to 1^-} f'(x) = \lim_{x \to 1^-} \frac{(5x-3)}{3(x-1)^{1/3}} = -\infty$$

f is Cont. at $x = 1$

then f has a cusp at $x = 1$

$f(1) = 0$

f has a V.T.L at $(1, 0)$